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ABSTRACT

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LEARNING RESEARCH AND DEVELOPMENT CENTER

PREALGEBRA STUDENTS' KNOWLEDGE OF ALGEBRAIC TASKS WITH ARITHMETIC EXPRESSIONS

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July 1984

Technical Report No. UPITT/LRDC/ONR/APS–16

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Prealgebra Students' Knowledge of Algebraic Tasks with Arithmetic Expressions

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Abstract

Knowledge about the structure of arithmetic expressions enables people to reason effectively about such expressions, including an ability to judge equivalence under transformations. This paper reports an empirical study of six middle-school children who judged the equivalence of three sets of three-term arithmetic expressions with an addition and a subtraction operator. Analyses of thinking-aloud protocols on this task reveal that the students (a) use several different methods to parse and judge the equivalence of such expressions, (b) sometimes use a different parsing or judging method with the same expression, depending on which expression it is compared against, and (c) are able to work with different conceptual interpretations of expressions. Additional results are provided about specific errors that were made and trends in the students' application of these methods. The results are discussed briefly along with three comments on their educational implications.

Prealgebra Students' Knowledge of Algebraic Tasks With Arithmetic Expressions

The general hypothesis motivating this research is that students will learn algebra more readily and with more understanding if they first learn algebraic concepts applied to arithmetic expressions, and can understand how these concepts are used to perform algebraic operations on arithmetic expressions. For example, we suppose that students would be better able to learn algebra if they already understood concepts such as commutativity and unary parsing (i.e., keeping the sign with the number) in the context of arithmetic expressions and could use them to perform such algebraic tasks as judging the equivalence of two arithmetic expressions. This hypothesis is particularly relevant to the educational goal that students should understand algebra as generalized arithmetic (Kieran, in press).

As a first step in exploring this general hypothesis, we assessed what prealgebra students know about the structure of arithmetic expressions. More generally, knowledge of structure is knowledge about the mathematical relationships among the elements of a mathematical expression. This knowledge is needed to perform mathematical operations on both arithmetic and algebraic expressions. For example, knowing how to interpret arithmetic expressions includes knowing that an operator only applies to the number immediately after the operator (when there are no parentheses), or that $a + b + c$ could be grouped as $a + b$ and c or as a and $b + c$. Knowledge of transformations that preserve equivalence includes knowing that $a + b$ equals $b + a$ and that $a - b$ does not equal $b - a$.

We presume that knowledge about structure is embedded in the performance of a variety of mathematical tasks such as interpreting expressions, computing the values of expressions, judging the equivalence of expressions, and performing transformations on expressions that preserve equivalence. We believe that characterizing knowledge about the structure of arithmetic expressions will provide a general way for describing students' knowledge of algebraic concepts and how to use them in specific algebraic tasks.

Structural knowledge need not be explicit to the student. What we are thinking of here is roughly comparable to the idea found in linguistics that we can generate grammatically correct sentences, that is, ones that obey the structure of the language, without necessarily knowing explicitly the structural rules being obeyed. By analogy, people can operate within the structural constraints of an arithmetic expression without explicitly knowing those constraints.

One important task that hinges on knowledge about the structure of expressions is the ability to identify and preserve equivalence. Equivalence is fundamental to mathematics. In algebra, all transformations on expressions (e.g., solving equations, or simplifying and factoring expressions) must preserve equivalence. In arithmetic, one must not change the values of numbers in the course of calculations.

The research reported here attempts to identify aspects of structural knowledge embedded in the performance of an equivalence-judging task by prealgebra students in middle-school (grades 6 - 8). The task involves judging the equivalence of three-term arithmetic expressions with addition and subtraction,

without computing any values.

We can infer knowledge about the structure of arithmetic expressions from performance on this task because it requires the students to interpret and compare transformed expressions. We assume their procedures must reflect aspects of their structural knowledge for two interconnected reasons. First, this is a novel task for these students--their instruction has not been inclined toward explicit, analytic examination of arithmetic expressions. Second, because this is a novel task, the students must use some understanding of the structure of arithmetic expressions as a guide to developing procedures.

For purposes of analysis, the process of judging the equivalence of arithmetic expressions can be viewed as functionally comparable to judging the semantic equivalence of natural-language sentences. Consider how people judge the equivalence of John gave the book to Mary to Mary was given the book by John. A simple model of this process holds that the two sentences are parsed into meaningful units, and the units compared semantically by a process that yields an equivalence judgment (Carpenter & Just, 1975). By analogy, a simple psychological model of how a person might perform this arithmetic task is as follows. First, a parse is made of the two expressions to be compared, identifying meaningful units. Then, the resulting units are used as arguments to a reasoning process that decides the equivalence of two expressions. This analogy reveals two psychological processes that might be involved in performing this task: parsing methods and judging methods.

Thinking-aloud protocols collected from students performing this task were analyzed to identify the categories of parsing and judging methods used. These two aspects of performance are taken to reflect students' knowledge about the structure of arithmetic expressions. The analysis also identifies the range of methods that the students use and some functional characteristics of the use of these methods, and errors that are made. The general theme of these results is that the students use several different parsing and judging methods, and that the parsing or judging of a particular expression sometimes depends on the expression against which it is compared. Several errors in parsing and judging, usually involving the subtraction operator, were included among these methods.

We aim ultimately to specify more clearly what is meant by "knowledge of structure of arithmetic expressions." We view the descriptions of the methods and their functional characteristics as an intermediate stage between the protocols and a satisfactory model, and believe these kinds of descriptions provide us with a foundation for developing a formal model of structural knowledge. Meanwhile, they are of interest for what they suggest about the nature of students' structural knowledge. In particular, the results support the hypothesis that students are able to reason with a number of concepts needed in algebra, but are missing some important constraints on this knowledge. The paper closes with a general discussion of these points and some of their educational implications.

Method

Task and Materials

The task was to judge the equivalence of three sets of arithmetic expressions without computing numerical answers (see Table 1). Each

expression had three terms and an addition and a subtraction operation except for one expression in the first set. Within each set of expressions the same numbers were used, but different numbers were used across sets. The first set had four expressions composed of small-digit numbers. The second set had four expressions composed of three-digit numbers. The third set had five expressions also composed of three-digit numbers. When they were presented to the children, each expression was written on a separate 4 in. (10.16 cm) x 6 in. (15.24 cm) card.

Insert Table 1 about here

We will refer to the sets respectively as the 12's set, the 947's set, and the 648's set. These expressions were selected for the following reasons. The 12's set serves in part as a check that the children can solve at least some equivalence-judging problems. All the numbers are small. Each expression started with the same number. This number was also the largest of the three terms in each expression. The second 12's expression, in the list in Table 1, had two additions in contrast to the other three that had one addition and one subtraction. This expression was included to check that the students had at least a minimal ability to do the the task. Also, the third and fourth expressions were included to offer the possiblity of inverting subtraction. In this set, only the first and fourth expressions are equivalent.

The 947's set had the largest number first in three of the four expressions. The positions of the addition operator and the subtraction operator are crossed with the positions of the last two numbers. This yields the first, second and fourth expressions. The third expression was varied by switching the position of the first and last numbers. In this set, the first and fourth expressions are equivalent as are the second and third. Two special features were designed into this set. The first two expressions in the 947's set in Table 1 were included because the subexpressions formed with the 685 and 492 are equivalent when considered in isolation, but are not equivalent when considered in the context of the whole expression. The last two expressions were included because their terms are in reverse order. That is, one begins with 685, has 492 in the middle, and ends with 947; the other begins with 947, has 492 in the middle, and ends with 685.

The 648's set was designed to not have the same first numbers, and by having five choices, the subjects could not try to second guess a symmetrical pattern as in the 947's set. The equivalent pairs in this set are: first and fourth, and second and fifth. The second and fourth expressions are in reverse order. They also offer opportunities to invert subtraction for a pair, or to insert implicit parentheses around the 873 and 597 and invert the subtraction with the 348.

Subjects

Six children, five sixth-graders and one eighth-grader, participated in the study. They were competent, probably above-average, prealgebra students. At the time of the interviews, near the end of the school year, they were working in the 8th grade

book of the Scott-Foresman textbook series. Many of them were going to enter a slowly-paced algebra course in the following school year.

As best we can tell, judging the equivalence of arithmetic expressions was completely novel to the students. They reported informally that they had never done such a task. Also, we examined the textbook series from which they were taught. In general, mathematics curriculum in America follows the textbook closely. Any deviation from the text is usually the omission of sections and not a supplement to the text (Freeman et al., 1983; National Advisory Committee on Mathematics, 1975). The curriculum is heavily oriented to calculating different expressions, with fractions, decimals, signed numbers, and with different operations. We note that the students never encounter a three-term expression with addition and subtraction. The only mixed operations are addition and multiplication and in these cases parentheses are always given. As you might imagine, the students are not given problems in which they are asked to judge the equivalence of two expressions.

Procedure

In individual, audio-recorded interviews, the students were first asked "what it means for two expressions to be equivalent." and then told that they would sort arithmetic expressions according to equivalence. The 12's set was presented first, followed by the 947's set, and then the 648's set, with the expressions presented in the descending order given in Table 1. To start, we presented the first pair of expressions in a set and asked the students to judge the equivalence of the pair. They were told not to compute, which they usually obeyed; and we generally discouraged numerical, but not ordinal, estimation. We also asked them to explain how they made

their judgment, and usually probed to clarify their answers. No feedback about correctness was given. The two expressions were placed together if they were judged equivalent, and separately if not. After each judgment, we presented another expression to compare against the ones already presented until all expressions in the set were judged. We then removed the set, recording the pairs of expressions judged equivalent, and proceeded to the next set. The recorded interviews were transcribed for the data analysis.

Results and Discussion

The major data analyses reported here examine the kinds of parses the students made, and the rules they used to judge equivalence of expressions. To be sure, we think analytic separation of the parsing operations from the equivalence-judging operations may sometimes distort the character of the actual process, but it provides descriptive summaries of important aspects of performance that cannot be seen readily from reading individual protocols.

Correctness of Performance

Before examining parsing and judging methods, let's first look at the correctness of the performance. These data do not play a central role in addressing the major questions, but they help establish a picture of the difficulty of this task for the students. We scored whether a student correctly or incorrectly judged the equivalence of a pair of expressions. Only those pairs on which a student explicitly commented were scored; uncommented pairs were unscored. A summary of this analysis for each subject is shown in Table 2.

Insert Table 2 about here

These data support the following conclusions.

1. Equivalence-judging is not an impossible task. The students produced correct answers for the comparisons in the 12's set. This perfect performance is not a result of the students calculating the answers. In some cases, they were asked whether they had calculated and they claimed no; in other cases where students computed these expressions, they made errors in calculation, supporting the contention that their perfect performance was not a consequence of surreptitious calculation.

2. On the other hand, equivalence-judging is not always an easy task to perform correctly. The students were correct only about half of the time on the 947 set which with pairwise comparisons is chance-level performance.

3. The comparisons were not differentially difficult for the students. The two students who made the most errors with the 947 set also made the most errors on the 648 set. Similarly, the two students who were most correct on the 648 set were among the best on the 947 set.

In sum, these data indicate that students can do this task, although not very well. Of greater concern, and to which we now turn, is whether there is anything systematic or regular in how students generate their answers.

Parsing Performance

The purpose of this analysis is to examine the structure or outcome of the students' parsings, showing the several kinds of parsings used in their equivalence-judging performance. The students'

explanatory statements, given after they classified a pair of expressions as equivalent or not, were analyzed to identify the parsing. Questions about how or why these parses were made can be considered once the basic phenomena are established.

Three general results are established here. First, the same pair of expressions can be parsed in different ways by different students. Apparently, a particular expression does not determine a particular parse. Second, the same expression, when compared against different expressions, is sometimes parsed in different ways by the same student. This suggests that the parsing of an expression can be context dependent. Third, some parsings violate the conventional mathematical interpretation of these expressions. Specific kinds of errors are indicative of the absence of important constraints in a student's understanding of the structure of arithmetic expressions. Evidence for these results is presented after a discussion of the observed categories of parses and the criteria used to classify a protocol as indicative of a particular parse category.

Criteria Used to Identify Parsing

Our analysis of the parsing performance is predicated on the following construal of parsing. First, we distinguish two senses of parsing. One sense refers to the units of parsing that the students used. The two major kinds of parsing units are: binary and unary. The other sense of parsing refers to the sequence in which these units are constructed to relate the terms of an expression. These two senses correspond to a phrase-structure parsing of a verbal sentence. The first sense corresponds to the units used in the parse tree; the second to the particular relations described among the units. In general we focus on the second sense, but, as will be seen, the two

are coextensive in some cases.

We used four categories of whole parses to describe the performance. A whole parse is one in which all the terms of an expression are related to each other. One category is, called binary, includes all parsings in which binary parse units were clearly used. A second category, called unary, includes all the instances in which a unary parse unit was used. The other two whole-parse categories are special sequences in which the focus is more on a global relation among all the elements rather than the units used. One is a left-to-right parse, the other is a backwards parse.

In addition to these categories of whole parses, four additional categories, reflecting mostly other parsing units, were needed to cover the observed performance. The need arose because sometimes students' protocols did not provide sufficient indication that they had examined and related all the terms of an expression. And in some cases, it seemed pretty clear that they had not.

The unit of the binary parse is a pair of terms connected by an operator. Thus, in a three-term expression there are three sequences in which whole binary parses can be constructed, though they do not always preserve equivalence. In the following example they do. In the expression a + b - c, the binary parses are: (a) a + b and then q - c, where q is the first binary pair; (b) b - c and then a + q, where q is the first binary pair; and (c) a - c and then q + b, where q is the first binary pair.

Although there are these different sequences in which binary units can be used to parse a three-term expression, we used a single category for all of these cases. It will be easier to explain why we do not differentiate the parsings in this category after we have discussed specific student performances. This discussion can be found in the section titled Across-set Parsing Performance.

Protocols statements were classified as evidence for binary parsing when students talked explicitly about intermediate results and used words such as then to talk about the next step. Here are two protocols that were classified as binary. They are meant to give a sense of the sort of protocol classified as binary. In the first protocol, Stella was comparing the expression 947 + 492 - 685 with the expression 685 - 492 + 947. She says:

Case 1: 947 + 492 - 685 685 - 492 + 947

947 + 492 would be these two together and then you subtract the 685.

It appears that she is taking 947 + 492 to be one binary unit, and then the result of that operation as part of a second binary unit with the 685. In the second protocol, Rudolf is comparing 648 + 873 - 597 and 873 + 597 - 648. He says:

Case 2: 648 + 873 - 597 873 + 597 - 648

Because this [648] and this [873] adds up to more than on the first one [873 + 597]. And then you'd subtract 579 [sic], which is less than 648.

It appears that he is taking the additive pair as one binary unit, and then composing the outcome of this operation with the last term of the expression to form a second binary unit.

The unit of the unary parse consists of a operator-number pair. That is, the unary parse unit treats the operators in an expression as a function that requires one argument (Weaver, 1982). By contrast, the binary parse unit treats the operators as function that requires two arguments.

The sequence of the unary parse is a term-by-term segmenting of the expression. This kind of parsing is common in algebra. There are two senses of unary parsing we want to distinguish here. The distinction hinges on the interpretation of the first number of an expression. On one view, the first number is treated as a starting quantity. Other quantities are added to or subtracted from this starting quantity. On another view, there is no starting quantity and any of the terms of an expression can be combined. Both of these interpretations are unary parses because the psychological units are an operator and a number. However, the first interpretation imposes a greater constraint on the operations performed on the expression.

Protocol statements were classified as unary parsing when students were comparing expressions by operator-number pairs, regardless of the order of their occurrence in the expression. This interpretation was buttressed by statements in which just the numbers were compared, and by statements of irrelevance of order. Here are two examples of protocols that were classified as unary parsing. They come from different students comparing the two expressions 12 - 2 + 6 and 12 + 6 - 2. Joanie said:

Case 3: 12 - 2 + 6 12 + 6 - 2

Well, you have the same numbers in the problem and you have the same signs. And you switch them around.

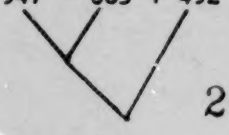
This student seems to have compared the numbers and the signs separately, and then notes that another order can be selected. Mary said:

Case 4: 12 - 2 + 6 12 + 6 - 2

You're taking away the same amount and you're adding the same amount. It's just a different order.

This protocol is classified as unary because the student is comparing amounts that are being taken away and added, and notes the irrelevance of the order in which this occurs.

In some protocols, students obviously made whole parses (i.e., they compared all the elements of the two expressions), but it is not possible to clearly distinguish whether a binary or unary parse was used. What seemed to predominate in these cases was the sequence used to parse and not the units that were used. In one such case, the special feature of the performance was that parsing proceeded from left-to-right, a common order in parsing arithmetic expressions. These cases were coded as left-to-right parsing. In this judging task, left-to-right binary and unary parses are functionally equivalent. Whether an expression is viewed as two binary steps, or as a string of unary operations, the same numbers would be added and subtracted in the same order if they were processed from left-to-right. To illustrate this point, consider the following example in which a student compared 947 - 685 + 492 with 947 - 492 + 685. If a left-to-right binary parse were used, then the expression would be parsed as follows:



If it were parsed with left-to-right unary parse, then it would be parsed in this way:

| 947 | | - 685 | | + 492 |

If a person were to compute these expressions, proceeding from left-to-right, using either parse, then the same outcome would be obtained. Rather than forcing a binary or unary interpretation on a protocol when there is not clear evidence for either, this third category was created. Cases are classified as left-to-right parses only when (a) they cannot be clearly assigned to a binary or unary parse and (b) the subject refers to the terms in the expression in left-to-right order.

Here is an example of a protocol that was classified as a left-to-right parse. Mary was comparing the two expressions just mentioned. The protocol started with the student judging them equivalent and then offering the following explanation:

Case 5: 947 - 685 + 492 947 - 492 + 685

S: Because in the first one, you're taking away more, but then you add the same amount you're taking away in the second one.

E: Here

S: You take away...

E: 685

S: ...685 and you add 492 in the first one.

E: Yeah.

S: And in the second one you take away 492 and add 685.

This protocol could be interpreted either as a binary or unary parse. On a binary interpretation, she first says that you take away more, thus treating 947 - 685 and 947 - 492 as binary units. She then says that you add the same amount you are taking away. The statement "but

then" suggests that she now takes 947 - 685 as a single term, the outcome of performing the subtraction operation, and has it serve as another binary unit with the 492. Similarly, in her later statements in which she describes the sequence of operations, one could impose a binary interpretation. On a unary interpretation, her initial description simply treats each of the terms as a unit and she discusses which units are being added and subtracted. The protocol does not provide strong evidence for either interpretation. What does seem stable, is that she is interpreting the expressions in a left-to-right manner.

The backwards parse involves interpreting an expression from right-to-left. Sometimes the operators are read right to left, but need not be. For example, in comparing 947 + 492 - 685 and 685 - 492 + 947, Mary said:

Case 6: 947 + 492 - 685 685 - 492 + 947

> They're just backwards. Well see, it starts out with 947 and you add 492 and you subtract 685, and then in [the other] you start out with 685, subtract 492, and you add 947. They're just opposite.

Mary has noticed that the order of the terms in one expression are opposite from the order of the terms in the other expression. She has used a left-to-right parse with the first expression and a backwards parse with the second expression. Such cases were classified as a backwards parse.

In addition to the four whole-parse categories just discussed, there were four categories of partial parses. These categories were created because in some cases it is difficult to decide whether a

student is parsing the whole expression. For some expressions, the students gave justifications that did not mention all the terms in the expression. In these cases, there is not explicit evidence to tell whether the students were applying a whole-expression parse as discussed above or quitting after comparing some portion of the pair of expressions.

One category involved another parsing unit: the operator. Protocols were classified into this category when the student only mentioned the operators in the pair of expressions. For example, in comparing 12 - 2 + 6 with 12 + 2 + 6, a student might simply note that the operators are different. A second category was an examination of a first binary pair in an expression. For example, the students might simply discuss the first binary pair in the preceding pair of 12's expressions. The third category covers one subject who mentioned only a term that was subtracted. This may be either a binary or a unary unit. The fourth category also involves another parse unit: the first number of a pair of expressions. This category covers one subject who only mentioned this unit in some protocols.

In some of these partial cases, subjects may have parsed the entire expression and only reported the critical portion that triggered their judgment. In these cases, with a plausible conjecture that the students are noting the equality of identical portions of two expressions, one can see that at least a left-to-right parse was made. For example, in the pair of 12's expressions the students may have noted that 6 was added in both expressions and so limited their comments to the point of difference. In other cases, such as the first-number category, it seems pretty clear that the students only

examined a portion of an expression to decide whether a pair was equivalent.

Within-Set Parsing Performance

Using the parsing categories just discussed, we attempted to determine the parsing of each pair of expressions that students compared. Table 3 contains the frequency of occurrence of the different parsings for the three problem sets.

Insert Table 3 about here

As might be expected, the majority of the parsings were classified as binary. However, there were also a number of other kinds of parsings. The frequencies probably reflect to some extent the particular structural characteristics of the selected expressions. Certain trends in the frequencies are discussed in the next section.

Protocols from each of the three problem sets will be discussed in turn to illustrate the three general results about parsing mentioned above. A large number of examples are given for three reasons. First, because a major purpose of the analysis is to establish the different parsing methods that students used, we want to show the protocols used to develop the parsing categories. Second, many of these examples are also discussed as judging performance. Third, examples of the three parsing results from the three problem sets establish that the results are not problem-set specific.

Parsing of the 12's set. In the 12's set there were six possible pairs to be compared; with six subjects, there are 36 possible comparisons. The subjects provided explicit justifications of their

answers on 19 pairs. The breakdown of these categories are shown in Table 3. There are not a remarkable number of differences in how the students parsed the expressions in the 12's set. Parsing performance on this set of expressions is of greater interest when it is contrasted later in the discussion with performance on the other two sets.

To illustrate the first result about different parsings for the same pair of expressions by different students, compare the performance of Rudolf and Stella when they compared the expressions 12 - 2 + 6 and 12 + 6 - 2. Rudolf's protocol gives evidence for a unary parse, while Stella seems to use a binary parse. Rudolf said:

Case 7: 12 - 2 + 6 12 + 6 - 2

> It doesn't matter whether you minus 2 before or after you add up 12 and 6. .... Because it doesn't matter whether the 6, it doesn't matter if you subtract the 2 before or after the 6.

It seems that Rudolf is interpreting each of the numbers as a separate quantity to be added or subtracted from 12. This parse is shown graphically in Figure 1.

Insert Figure 1 about here

By contrast, Stella's protocol was:

Case 8: 12 - 2 + 6 12 + 6 - 2

> It's just mixed up because it's still 12 + 6 as it is here, and then take away 2.

In this case, Stella seems to have taken 12 + 6 as a unit, and compared it with the other expression, and then considered the remaining number as a pair with the 12 + 6 unit. Her parse is shown

graphically in Figure 1. These protocols illustrate the point that the same pair of expressions can be parsed in a different manner in the course of judging their equivalence.

There were no clear cases of different parses for the same expression or mathematically incorrect parses in this set. All of the partial operators that appeared here seem to be cases in which the student simply reported the point of difference between two expressions. Each of the cases could have been reinterpreted as a binary or left-to-right parse if one assumes the equivalence of the reflexive portions were judged. For example, in comparing 12 - 2 + 6 with 12 + 2 + 6, a couple of subjects simply noted that the first pair was different.

Parsing of the 947's set. For the 947's set, there were 36 possible comparisons. The subjects provided explicit justifications of their answers on 22 pairs. The breakdown of the different kinds of parsings are shown in Table 3. The total equals 23 because one student used two different parses for one pair.

Two examples illustrate the first result about different parses. In the first example, Rudolf compared 947 - 685 + 492 with 947 - 492 + 685. He declared they were equivalent and explained:

Case 9: 947 - 685 + 492 947 - 492 + 685

> Because it doesn't matter if it's 492 or 685 here, as long as it's minus 947. And as long as it's plus there. ... It doesn't matter that these two are switched around. [E: The 685 and the 492?] Yeah, as long as the minus is right there, and the plus is in between those two.

Rudolf's explanation suggests that he made a binary parsing of the two

expressions. This parsing is shown in Figure 2. By contrast, Joanie seems to use a left-to-right parse. She explains that they are equivalent as follows:

Case 10: 947 - 685 + 492 947 - 492 + 685

Because on this one you subtract 685 but you add 492 back. And on this one you subtract 492 but you add 685 back.

Indeed, whether Joanie is using a binary or unary parse, it is distinctly different from Rudolf's parse for the same pair of expressions. Joanie's parse is also shown in Figure 2.

Insert Figure 2 about here

In the second example, Mary and Stella were comparing the expressions 685 - 492 + 947 and 947 + 492 - 685. Using a binary parse, Stella judged this pair to be equivalent. Her protocol is repeated from Case 1.

Case 1: 685 - 492 + 947 947 + 492 - 685

Because it's just reversed, 947 + 492 would be these two together and then you subtract 685.

This parse is shown in Figure 3.

Insert Figure 3 about here

Mary also judged that these two expressions were equivalent, but she used a right-to-left or backwards parse. Her explanation from Case 6 is repeated. The parse is shown in Figure 3.

Case 6: 685 - 492 + 947 947 + 492 - 685

[They] are the same, they're just backwards. [E: I'm sorry,

what's backwards?] Well see, it starts out with 947 and you add 492 and you subtract 685, and then in [the other] you start out with 685, subtract 492, and you add 947. They're just opposite.

Two examples are presented to illustrate the second result that the same expression can be given a different parsing by the same person when compared with different expressions. In first example, Nick was comparing 947 - 685 + 492 with 947 - 492 + 685 and with 947 + 492 - 685. Nick's judged the first pair to be equivalent. His protocol was:

Case 11: 947 - 685 + 492 947 - 492 + 685

They just rearranged the numbers. [E: Which numbers?] 685 and 492.

It appears that Nick has used a binary parse on these two expressions. It is comparable to the parse shown for Rudolf in Figure 2. Nick judged the second pair to be equivalent as well. However, his parsing of the first expression now changes. His protocol was:

Case 12: 947 - 685 + 492 947 + 492 - 685

Because you're subtracting 685. .... It'd just be the same as doing these two [947 + 492] and subtracting all that [685]. [E: The same as, 9-] This plus that... [E: + 492] ...minus 685.

The parse is shown in Figure 4. Nick seems to have given a left-to-right binary parse of 947 + 492 - 685. He then uses the resulting parse as a template against which to compare the 947 - 685 + 492. He notes there is a 947 and a + 492, which he takes as a comparable binary unit, and then subtracts that by 685.

Insert Figure 4 about here

In the second example, Stella compared the same first pair as Nick and parsed it the same way as Rudolf and Nick (see Cases 9 and 11 and Figure 3). She then went on to compare 947 - 685 + 492 with 685 - 492 + 947. Now she uses a binary parse on the two expressions with a left-to-right sequence. She judged the two expressions to be not equivalent and explained:

Case 13: 947 - 685 + 492 685 - 492 + 947

> Because 947 plus the remainder of that [685 - 492] would probably be more than 947 take away 685, plus the remainder of that [492 added to the result of 947 - 685].

She discusses the second expression first, saying that if you added 947 to the result of 685 - 492, that it would be more than 947 - 685, then adding in the remaining 492. Her parsing is shown in Figure 5.

Insert Figure 5 about here

These two examples show that the same expression can be parsed in a different way according to the expression against which it is compared.

Concerning the third result, there are a number of parsings in the 947 set that violate mathematical conventions. Some have already been presented. The parsing that Stella, Rudolf, and Nick made of 947 - 685 + 492 and 947 - 492 + 685 was one such error (see Cases 9 and 11). For purposes of labeling, we call it the "binary error." The error is that the students made an additive pair a unit and applied the minus operator to the sum of the additive pair, rather than the first term of the pair.

The other notable error was the right-to-left or backwards parse. Two of the six subjects noted that 685 - 492 + 947 and 947 + 492 - 685 were equivalent because they were just backwards. One student, Mary (Case 6), proceeded to read both expressions with the proper signs. Exact interpretation is not possible, but she does not seem highly concerned about what numbers the operators are assigned to. The other subject who made this mistake, on this and other pairs was more explicit. He discussed how one could read an expression in either direction and that the sign preceding the number (depending on which direction one was going) would be assigned to the number.

There is one partial parse worth noting. Joanie was concerned about the number that the expressions started with. In the 12's set, all the expressions started with the same number. In this set, one of the expressions does not start with the same number. She first judged 947 - 685 + 492 and 947 - 492 + 685 to be equivalent (see Case 10). She then encountered 685 - 492 + 947. She said:

Case 14: 685 - 492 + 947 947 - 685 + 492 947 - 492 + 685

That goes in a separate pile because the number you start out with is lower than in this pile.

Later when she was discussing how she did this set of expressions, she said, "See if, it depended on where the signs were with the numbers. But the first number always had to be the same. Because that's the starting element."

Parsing of 648's set. For the 648's set, there are 10 possible pairs of expressions to compare. One of the subjects' statements was uninterpretable with respect to the parsing he used, so the data from only five subjects were examined, making a total of 50 possible

responses. There were 33 interpretable responses for explicitly compared pairs. The breakdown of the parsing is shown in Table 3.

Two examples are given to illustrate the first result that the same pair of expressions are parsed differently by different students. In the first example, students were comparing 648 + 873 - 597 with 873 + 597 - 648. Stella used a binary parse on this pair. She decided that this pair was equivalent and explained:

Case 15: 648 + 873 - 597 873 + 597 - 648

> If you added 873 + 597, you would get a smaller number than 648 take away [sic] 700, or 873. And then you take away a larger number and then a smaller number.

The parse for this expression is shown in Figure 6. It seems that Stella was examining the the first binary pair and then considering that pair as a unit with the last number in the expression.

Insert Figure 6 about here

By contrast, Joanie also judged the same pair of expressions to be equivalent, but she used a left-to-right parse that seems closer to unary than binary. Her protocol was:

Case 16: 648 + 873 - 597 873 + 597 - 648

> You added this, the highest number plus the lowest number, and you subtracted the middle number. In this you added the highest number, the middle number plus the highest number and subtracted the lowest number.

It seems that Joanie is considering the number in a left-to-right manner, but she takes the whole expression and their interrelations in her parsing rather than two binary units as we find in Stella's

protocol. The structure of Joanie's parse is shown in Figure 6.

In the second example, we want to show three different ways that students parsed the expressions 873 + 597 - 648 and 648 - 597 + 873. These were the only three observations obtained on this pair, reinforcing the point that different students sometimes make different parses on the same pair of expressions. In the first case, Nick used a left-to-right parse in examining the two expressions. He first claimed that the two expressions were different, describing the latter as:

Case 17: 873 + 597 - 648 648 - 597 + 873

> Because you're subtracting these two and adding this and it would come out to around...[E: Subtracting which two?] These two [E: The 648 and 597?] Yeah. And you're adding 873. So it would come out to be... I want to check this.

Nick proceeded to estimate the values of the two expressions and subsequently decided that they were about the same. This left-to-right approach is not exactly driven by an analytic approach to comparing this pair of expressions, but it provides a contrast to the other two approaches that were observed. Rudolf approached this set of expressions with a binary parse. He decided that these two expressions were not equivalent and explained as follows:

Case 18: 873 + 597 - 648 648 - 597 + 873

> Because these two [873 + 597] added up are always higher than this [648]. So they wouldn't be negative numbers. But these two [597 + 873] added up and then subtracted from this [648] would be a higher negative number. Because these two are higher than this. And if you subtract them from that....

Rudolf's parse, shown in figure 7, shows how he made binary pairs with

the addition operators and then composed the result into a binary pair with the subtraction operator. Mary judged this pair of expressions to be equivalent. Her parse was a backwards parse, shown in Figure 7. In her protocol she refers to the two expressions by labeling numbers that were on the cards with the expressions. The protocol went:

Case 19: 873 + 597 - 648 648 - 597 + 873

Because it starts out in two it goes 873 + 597 - 648, in four it's 648 - 597 + 873. It's just the rev-, they're backwards, and it would be the same.

Insert Figure 7 about here

Two examples are given to illustrate the second result that the parsing of a particular expression is affected by the expression against which it is being compared. In the first example, Nick is comparing 597 - 648 + 873 with 648 + 873 - 597 and 648 - 597 + 873. Nick judged the first pair to be equivalent to each other, explaining:

Case 20: 597 - 648 + 873 648 + 873 - 597

You can just add these two and subtract 597.

Here it seems that Nick has made a binary parse with the two additive pairs, and then made another binary pair with the subtractions. This parse is shown in Figure 8. For the second pair, which Nick judged immediately after the first pair, he said they were equivalent and explained:

Case 21: 597 - 648 + 873 648 - 597 + 873

Because this number 648, it's just being changed around as far as I understood it.

We take this to mean that Nick is inverting the first pair of numbers

in 597 - 648 + 873, thus making it identical to 648 - 597 + 873. The parse for this expression is shown in Figure 8 as well. One can see that the parse made on 597 - 648 + 873 has changed depending on which expression it was being compared with.

Insert Figure 8 about here

In the second example, Joanie was comparing 648 + 873 - 597, 873 + 597 - 648, and 597 + 648 - 873. In comparing the first pair, which she judged equivalent, she used a left-to-right parse in which she compared the three numbers at once. Her protocol, given in Case 16, is repeated here.

Case 16: 648 + 873 - 597 873 + 597 - 648

You added this, the highest number plus the lowest number, and you subtracted the middle number. In this you added the highest number, the middle number plus the highest number and subtracted the lowest number.

In comparing these two expressions she is coordinating the relations between the three numbers, noting which numbers were being added and subtracted. She did a similar comparison of magnitudes in comparing the first two expressions with the third expression. But now there is a distinctly binary parse. She says that the third expression is not equivalent:

Case 22: 597 + 648 - 873

Because this makes a smaller number. [E: The first two terms?] Yeah. And then you subtract a greater number.

She is no longer coordinating the three numbers in one structure, instead focusing on the first pair and then using that as a unit to

compose into another binary pair with the last digit.

Concerning the third result, the same errors we observed for the 947 set were also observed for the 648 set. In particular, there were a number of cases in which the students formed binary pairs with the additive pair and subtracted the sum from the first number in the expression. There was also an instance of the backward parse. In addition, we observed cases in which the students were inverting a binary subtraction pair. This occurred both for a single binary pair and for a binary pair composed of an additive pair and a subtraction. Examples were shown above in Nick's protocol (see Figure 8).

A final note about a partial parsing that occurred in this set of expressions. When Joanie encountered 597 - 648 + 873 she said, "This is impossible. You can't subtract 648 from 597 unless you're using negatives and positives." This case shows that sometimes students do not need to parse all the terms of an expression to decide on its equivalence.

Across-Set Parsing Performance

This section discusses some of the general characteristics of the parsing across the three sets of expressions. These characteristics are taken to be primarily a consequence of the structure of the expressions, rather than some general parsing capability of the students. The problem sets were not designed to provide a comprehensive coverage of the kinds of expressions that one might encounter, so it is not reasonable to draw any conclusions about the frequency of different kinds of parses. However, there are two notable trends in the parses observed when comparisons are made across the three problem sets.

First, there were different distributions of parses across the three problem sets. These differential distributions are supportive of the point that content of the expressions can affect the kinds of parses that one observes. On the 12's set there was a greater occurrence of unary parses, and there were comments about the subtraction operator that we did not see in the other two sets. We suspect that the unary parses on the 12's set may be a function of (a) numbers that are sufficiently small to enable the students to form a conceptual structure relating the effect of adding and subtracting each number to the value of the expression, and (b) expressions beginning with the same number, which was also the largest, so that it was possible to simply evaluate the effects of adding and subtracting the other two numbers to the first number. This does not mitigate the fact that the students were doing a unary parse. It just raises a possible boundary condition on the generality of their unary parsing abilities. For the 947 set, no unary parses were observed and there was roughly an even split between clear binary and non-binary cases. For the 648 set, the majority of parses were binary.

When one examines the sequences in which binary parses were conducted, then it is apparent that there is no predominating bias. This point is important to notice because the cases presented above are consistent with an induction that says subjects prefer to form their first binary unit with the plus sign. In fact, most of these binary parses seem to follow from a number of different structural features of the expressions, rather than a strategic preference of the subjects. Of the 11 binary parses observed for the 947's set, 4 of them followed a left-to-right sequence on expressions that also happened to have a minus sign as the first binary unit. Five formed

the first binary unit with the plus sign, without following a left-to-right sequence; but these pairs also had the same numbers, hence a likely unit for students to form. Only 2 of the 11 binary parses formed the binary unit with the plus sign first, with both numbers not the same, and did not follow a left-to- right sequence. Comparable results were obtained for the 648's set. Of the 22 binary parses, 8 of them followed a left-to-right sequence in which the plus sign happened to be first in the expression. Three followed a left-to-right sequence in which the first binary unit was addition in one expression and subtraction in the other. Four formed the first binary unit with the plus sign, but these pairs also had the same two numbers. Six formed the binary unit with the plus sign first, with both numbers not the same, and did not follow a left-to-right sequence, and 1 formed the binary unit with the minus sign first and did not follow a left-to-right sequence.

Second, there appears to be some weak trends regarding the parsing used by individual students across the set of expressions. Mary, Joanie, and Nick seemed to follow a more left-to-right and unary approach. Stella and Rudolf seemed more oriented to binary parses. Only Stella seemed to consistently follow a single kind of parsing, the other students usually had a mixture. A particularly striking case was Rudolf who used unary parsing for the 12's set, and consistently used binary parsing for the other two sets. Moreover, each of his binary parses for the 947's and 648's formed the first binary unit with the plus sign. He was the student who produced the two cases for the 947's and the six cases for the 648's in which the plus sign was the first binary unit, with both numbers not the same, and did not follow a left-to-right sequence. These trends indicate

that individual students may be able to work with different kinds of parses, but have preferred parses that they work from in evaluating a set of expressions.

Judging Performance

There is a minor theme and a major theme developed in the results about judging methods. The minor theme is that the students do not seem to have well-developed strategies for attacking the equivalence-judging task. They did not seem to have algorithmic approaches nor did it appear obvious to them how to proceed. This is not surprising given that this is a novel task for them. The major theme is that, despite the minor theme, they were able to develop and use a number of methods. Like the parsing analysis, the primary concern is to identify the existence of the different methods with which the students judged the equivalence of these expressions. These methods are taken to reveal the student's knowledge of the mathematical structure of the arithmetic expressions, and provide evidence that the students may have the capacity to work with conceptual ideas found in their judging rules.

Evidence for the minor theme comes from the fact that most of the students encountered pairs of expressions for which they first asserted that they could not tell whether they were equivalent or not, or produced one answer. The students then proceeded to find a method, or switched their answers after closer examination of the pair of expressions, usually accompanied by a statement of "Well wait," which is taken as a conversational signal that a new idea has appeared. Here is a particularly good example. Rudolf, one of the more accurate students, was comparing 12 - 2 + 6 and 12 + 2 + 6. He had determined analytically that these two expressions were not equivalent. However,

he needed to compute the answers to reassure himself that his analysis was correct. This example is an illustration of the claim that the students do not have well-developed strategies and intuitions about the judging task.

Additional evidence comes from some of the judging methods discussed below that were noted as possible default methods. We presume these methods reflect the lack of developed methods. However, the analyses presented below show that the students have sufficient knowledge to develop a number of approaches.

The analysis of the judging performance will be developed in three parts. The first part discusses the judging methods used for the 947's and the 648's. As part of this discussion, some general considerations about the variety of judging methods will be introduced. The second part discusses the judging methods for the 12's. These are discussed separately primarily because the data underdetermined the classification of the protocols into the classes of methods developed in the first part. The third part presents some general results about the use of these judging methods across and within students.

Judging Methods for 947's and 648's

For each judging method, we will describe the method, present the criteria for classifying a protocol as indicative of this method, and discuss some illustrative examples of performance. The observed judging methods can be divided into analytic approaches and non-analytic approaches. Further divisions within these two general approaches will be addressed in turn. We discuss the analytic approaches first.

Analytic judging approaches. Analytic judging approaches provide answers that are necessarily true. These approaches can be divided into number-independent methods and number-dependent methods. We will discuss the number-independent methods first.

As their name implies, number-independent methods do not use the values of the numbers in the expressions to judge equivalence. The necessary truth of the number-independent judging methods is grounded on the mathematical axioms and theorems commonly found in algebra. We will refer to these axioms and theorems collectively as properties. The students use these properties in the process of judging the equivalence of elements in a pair of expressions.

Before we discuss these methods further, we would like to introduce the concepts we used to analyze these methods as well as the other observed methods. The judging task is a procedural task that must obey the general constraint of equivalence. Specific judging rules are applied within the constraints of higher-order principles (Greeno, Riley, & Gelman, 1984). A primary aim of our analysis of performance on this task is to analyze the protocols in such a way that it reveals, to some extent, the structure of methods used. We describe higher-order principles informally to give a sense of the organization of performance, and then describe specific realizations of a principle that the students used. We do not claim that the students are conscious of these principles in the form presented, only that this is a possible organizing principle that underlies and rationalizes what they are doing. Also in classifying performance, we have not separated the correct uses of mathematical properties from the incorrect uses. This is consistent with the concern to examine

the structure of the methods, and not the correctness of the content. We will however discuss the kinds of errors that occur, many of them should be correctable without much difficulty.

For most of the judging methods there is a version based on a unary parsing of the expressions and another based on a binary parsing. The general principle that organizes the unary parsing for the number-independent methods is that two expressions are equivalent if the same terms are added and subtracted, regardless of their order in the expression. This principle can be realized in specific procedures that compare the terms and operators between two expressions to determine whether the general principle has been satisfied. The general principle that organizes the binary parsing for number-independent methods is that two expressions are equivalent if the binary pairs in the expression are equivalent, where one binary pair can serve as a term in another binary pair. A number of properties can be used to determine the equivalence of binary pairs. Here is a list of some of the properties that the students seem to use:

$A = A$	Reflexive
$A + B = A + B$	Reflexive
$A + B = B + A$	Commutative
$A - B = A - B$	Reflexive
$A + Q = A + Q$	Reflexive, where Q is a binary pair
$A + Q \neq A - Q$	Where Q is a binary pair
$A + Q = Q + A$	Commutative

The discussion up to this point has described a set of forms within which equivalence judging can proceed. This is a structural level of analysis. We now want to mention briefly two possible psychological implementations that would embody these structures. One is syntactic; the other is semantic. (See the General Discussion for a speculation on a third possibility.)

Syntactic methods do not use the values of the numbers, one simply notes whether they are the same or different. The unary method would be realized syntactically by a procedure that determines whether the same numbers (or symbols) are being added and subtracted together. This is the method commonly used in algebra to judge the equivalence of two expressions. The binary method would do the same thing, but would use pairs, and the properties described above to determine equivalence.

Semantic methods interpret the numbers in the expressions as representing quantities of different relative magnitudes and use these magnitudes as part of judging procedures. Semantic procedures that realize the unary and binary methods are comparable to their syntactic counterparts, processing the terms of the expression in the same sequence. They differ from the syntactic approach because they view numbers are viewed as standing for quantities, and decisions are made using rules about operations on quantities.

In a unary version of a semantic method, a number that is common to the two expressions is taken as a "starting" number. In the 12's and the 947's problem sets, this was usually the largest absolute value and the first number in the expression (except for 685 - 492 + 947). The students can then check to see if the same

numbers are being added and subtracted to this start number. They can verify that the two expressions are equivalent if the same quantities are being added and subtracted. The binary version of this method compares binary pairs, checking that the same magnitudes are being added and subtracted.

The preceding discussion has introduced the concepts that will be used to discuss the judging methods. Rational inspection shows that any method could be realized in a unary or binary form and with a syntactic or a semantic process. In practice, not all of these possibilities are common, and the available data are not always adequate in general for making such determinations. These descriptive concepts are used to the extent that the distinctions can be made in the observed protocols.

Turning now to examine specific cases, only binary, syntactic methods were observed for the number-independent methods. No unambiguous cases of unary processing or semantic processes were observed for number-independent solutions. Table 4 shows that 14 cases of number-independent judging were observed as well as the frequency with which the other methods were used.

Insert Table 4 about here

Three criteria, each being sufficient, were used to classify protocols as indicative of a syntactic, number-independent method*. One criterion was statements in the protocol like "just rearrange the numbers" or "just change around the numbers." Such statements are taken to indicate that the students were looking at the syntactic

relations of the numbers. The procedures that carry out these operations do not use the values of the numbers. For example, Rudolf judged that 947 - 685 + 492 was equivalent to 947 - 492 + 685. His protocol was given above in Case 9, and repeated here. He explained:

Case 9: 947 - 685 + 492 947 - 492 + 685

I couldn't tell what the answer to this problem was, I just know it. It doesn't matter that these two are switched around. [E: The 685 and the 492.] Yeah.

In this case, Rudolf seems to follow a form of $a + b = b + a$. Then if these two equivalent binary pairs are represented by q, then $a - q = a - q$.

A second criterion for classifying a protocol as a syntactic, number-independent parse was statements of orders of operation. For example, Nick judged 648 + 873 - 597 to be equivalent to 597 - 648 + 873.

Case 20: 648 + 873 - 597 597 - 648 + 873

You can just add these two and subtract 597.

While this statement is telegraphic, it is interpreted as asserting that 648 + 873 is equivalent in both expressions, and then the same number is being subtracted from the equivalent binary pair. His statement was given immediately and his use of the word just is taken to indicate that he is manipulating the expressions in terms of their algebraic relations. In other words, the values of the numbers in the expressions are not used (except for noting their identity).

*As noted above, it is possible that these judgments could have been made semantically and reported in a syntactic manner. However, both the rapidity of the judgments and the fact that the students already knew how to recognize equivalence by the commutative property leave us satisfied with this projective interpretation.

A third criterion was statements about rearranging the number in certain orders. Stella also judged 648 + 873 - 597 to be equivalent to 597 - 648 + 873 explaining:

Case 23: 648 + 873 - 597 597 - 648 + 873

If the 648 were placed first and then 873 and 597 would be exactly the same.

Her statement is also telegraphic, and she describes a transformation that would make the two expressions identical. We take the telegraphic statement to indicate that these decisions are being made syntactically rather than reasoning with values of the quantities.

Another method classified as a syntactic, number-independent method was the backwards method. The principle here is roughly that two expressions are equal if their terms are in reverse order and the same operators are present. To some extent, this could be considered a unary method. The criterion for classifying a protocol as indicative of a backwards judgment was unambiguous. The students stated this explicitly. They did not give a rationalization for this judgment. Indeed, their protocols suggest that it was obvious to them, which is consistent with a syntactic interpretation. An example was presented above in Case 6 of the parsing analysis. This approach is included as analytic because the students are using some syntactic features as a guide for deciding equivalence, and these features take the form of a general rule that is applied with consistency.

The analytic, number-dependent methods use the semantics of ordinal values to decide whether a pair of expressions are equivalent or not. This system provides the necessary truth for these methods. It is number-dependent because one must have the values of the terms

of the expression in order to use this system. The general principle that organizes this performance is the idea that if one expression has a larger value relative to the other, then they cannot be equivalent. This determination could be made with binary or unary parses. No analytic performance was observed that could be classified as unary, so only the binary cases will be considered.

In this method, the students use the semantics of more and less in a direct, analytic manner, deciding equivalence according to the consequences of adding more or less to a quantity. The kind of rules used here are of the form: more - less > less - more. There were two main instances in which this method was used. The first was in the comparison of 597 + 648 - 873 with both 648 + 873 - 597 and 873 + 597 - 648. This accounted for 8 of the 14 observed instances of this method. Here is Joanie's protocol in which she judged the first expression to be less than the other two expressions, as did the other subjects.

Case 22: 597 + 648 - 873 648 + 873 - 597 873 + 597 - 648

This makes a smaller number. [E: The first two terms?] Yeah. And then you subtract a greater number.

Protocols were classified as indicative of an analytic, number-dependent method if they contained statements that compared the magnitudes being added and subtracted between the two expressions and these comparisons could be made analytically. Thus, in Case 22, the additive binary pair in the latter two expressions had one number in common with the additive binary pair in the first expression. However, the other number (873) in the additive binary pair in the latter two expressions was larger. Therefore, one can determine that the value of the additive binary pair for the first expression is

smaller than the value of the additive binary pairs for the latter two expressions. Finally, one notes that a larger amount is being subtracted from the additive binary pair in the first expression compared to the other two expressions. This permits the use of the rule described above.

The other main instance of a number-dependent method involved the rule that positive numbers are greater than negative numbers. This accounted for 5 of the 14 observed instances. For example, Rudolf (Case 24) asserted that 648 - 597 + 873 was not equivalent to 648 + 873 - 597, 873 + 597 - 648, or 597 + 648 - 873 because "None of the others would be negative numbers and this one would be." When asked to explain how he decided this, he explained how the first two added up are more than the number subtracted and how in the negative one, the two numbers added up were more than the 648. Despite the incorrect parsing of the first expression, his judging method is an analytic rule, based on the relative values of the numbers. Protocols were classified as indicative of this method when the students asserted that one expr ssion was negative and could not be equivalent to the other expressions.

Non-analytic approach. Having examined the number-independent and the number-dependent, analytic methods of equivalence judging, we now turn to the non-analytic approach. Non-analytic methods produce answers that do not have a logical system that guarantees the necessary truth of the conclusions*. There were three general methods

*Please note that we only apply the analytic concept to systems based on logics generally recognized as analytic. It may be necessary at some point to augment the analytic concept with a category of "personal-analytic" methods. That is, methods based on a consistent system of logic that is not generally recognized as analytic.

in the non-analytic approach. They are semi-analytic, compensation, and difference. These methods are primarily number-dependent. The semi-analytic and some of the difference methods seemed to function as default methods that the students used when they could not find a more analytic approach. The semi-analytic and the compensation methods were for the most part semantic because students use the relative magnitudes of the quantities and compare them in terms of more and less. These two methods are distinguished by the manner in which the semantics of relative magnitude are used in determining the equivalence of a pair of expressions.

In the semi-analytic method, the students set up the proper relative-magnitude relations between the subexpressions of an expression, but there are not analytic rules based on the ordinal semantics that can use these relations to produce a decision. Instead, the students seem to engage in a sort of estimation of these ordinal quantities. Thus, while the answers are close approximations, they leave the students with uncertain answers that are not analytic.

Here are two examples. The protocol for the first example is given above in Case 13. It seems that Stella is thinking something along the lines that a small quantity added to a large quantity might be a little larger than the two medium-size quantities being added. In another case, Joanie asserted that 648 - 597 + 873 and 648 + 873 - 597 were equivalent because:

Case 25: 648 - 597 + 873 648 + 873 - 597

This [648 + 873 - 597] you add, you get a high number and you only subtract a little. And this [648 - 597 + 873] you add a lot. So I think they should be equivalent.

It appears that Joanie is making an estimation of the sort that large minus small is equal to small plus large.

Two distinctive characteristics were used to classify protocols as semi-analytic. Contrary to other methods, the students usually hedged their assertions with statements like the two expressions "might be the same," "probably get the same thing," "might make up for the loss," or "might end up to be this." Second, the students did not compare the relative magnitudes (i.e., larger or smaller) of subexpressions between the pair of expressions as they did with the analytic number-dependent method. Instead, they talked about the relative magnitudes of the numbers within each expression, such as "add a high number," "subtract a little number," and "get a low number." Finally, although not a criterion for classification, in many of these cases, students expressed uncertainty about their answers by statements like "don't know exactly," and "can't really tell." This is consistent with the fact that this method does not provide an analytic solution.

In the compensation method, students use the semantics of ordinal quantities as a cue to a more general principle that asserts equivalence. This higher-order principle went something like this: "Two expressions are equivalent if both sides get comparable treatment. If a bunch of opposites are present, then maybe it will all balance out." This general idea takes a number of different forms and we have not found a general description to characterize them. For now, we will have to communicate this general idea by examples. Protocols were classified as indicative of a compensation method if the students asserted that the pair of expressions were

equivalent, and if they talked about the quantities involved in terms of more and less or larger and smaller, and talk about how one operation was performed in one expression and a different operation was performed in the other expression, but that it all comes out the same. The character of the last step was such that the students seemed to defer to the equivalence judgment of the principle rather than using the information about the relative magnitudes directly.

The compensation method seems robust. Four of the five subjects used a compensation method at least once. Also, it is possible to find contrasts that are suggestive of binary and unary methods and of semantic and syntactic methods. In the unary version, the students used the idea that some quantities were being added and subtracted to a start quantity, except now they did not pay attention to the fact that different quantities were being added and subtracted. Joanie judged that 648 + 873 - 597, and 873 + 597 - 648 were equivalent because:

Case 16: 648 + 873 - 597 873 + 597 - 648

> You added this, the highest number plus the lowest number, and you subtracted the middle number. In this you added the highest number, the middle number plus the highest number and subtracted the lowest number. See you subtracted less because you added more. You subtracted more because you added less. It seems confusing.

One can see that in this protocol Joanie operates with the relative magnitudes of the numbers. The two sentences before the last one are what we take to be indicative of a compensation method. Her reasoning seems to follow an argument along the lines of: addition and subtraction are opposite, these numbers are more and less, so in some

sense are opposite. If you subtract less in one expression, then you should do the opposite in the other expression, that is, you should add more. Similarly, if you take away more in one expression, then you should add less in the other. Because we are compensating for both numbers being added and subtracted, then it should balance.

We take this to be a unary method, in which the 873 is serving as the start number and Joanie is examining the effects of adding and subtracting the 648 and the 597. Also note that her statement about how it is confusing is consistent with the claim that she is not using an analytic method, nor using the magnitudes of the quantities directly. Joanie did not make comparable statements when comparing other expressions. However, just because this is confusing does not mean that she is uncertain about this method. The interviewer told her that she could change her mind if she wanted, but Joanie maintained that they were equivalent.

Stella compared the same pair of expressions with a compensation method, but used a binary version. In such a method, one notes a difference in a binary pair and then a compensating difference in the other binary pair. Her protocol was:

Case 15:

> If you added 873 + 597, you would get a smaller number than 648 take away [sic] 700, or 873. And then you take away a larger number and then a smaller number. So I think it would be.

Stella made several slips in which she substituted take away for add or add for take away, as did some of the other students. No significance is attached to it here. In this protocol, she compared the relative magnitudes of the numbers, noting that the first binary

pairs would not be equivalent, but then seems to use a compensation rule of the form: Larger minus smaller equals smaller minus larger. The compensation schema here seems roughly as follows. The description of the two start sums are semantically opposite, and one is doing the same operation of these opposites with another set of opposites, so it should all balance. This protocol was classified as compensation rather than semi-analytic because we presume that Stella does not believe that larger minus smaller equals smaller minus larger and that instead she is using the meanings of the quantities as a cue for a more general principle rather than as inputs to a procedure that needed this information to make a decision.

The syntactic-semantic contrast in the compensation method is illustrated on the pair of expressions 947 - 685 + 492 and 947 - 492 + 685. Joanie seemed to use a syntactic method. Her protocol is given above as Case 10. The compensation schema here seems to be of the following sort. Addition and subtraction are opposite. Number A and Number B are different. The start numbers are the same. If I take away A and add B, then on the other one I should add B and take away A, and it will all balance out. Her use of the word "but" suggests that she is using the compensation idea. In particular she seems to use this word as if to say, "I am starting by doing something that would seem to make the two expressions non-equivalent, but then I do something that compensates for the differences and returns them to equivalence." This protocol is described as syntactic because she does not seem to be using semantic ideas of more and less to select the compensation rule. Instead, she seems to simply note that different numbers are being added and subtracted.

By contrast, on the same pair of expressions, Mary seems to use a semantic approach. Her protocol was:

Case 5: 947 - 685 + 492 947 - 492 + 685

Because in the first one, you're taking away more, but then you add the same amount you're taking away in the second one.

This protocol seems to use a comparable schema as the one described for Joanie's protocol, and Mary even precedes her description of the second step with the word but. The important difference is that Mary's schema seems to operate on the idea that compensating relative quantities are being used, rather than compensating differences that Joanie was using. In this particular case, there is no behavioral difference between these two methods. However, we note the existence of this possible difference in the psychological realization of the compensation method because it helps clarify what might be the difference between rote and meaningful performance. This issue is considered briefly in the General Discussion.

While we have just argued that a person can come to a compensation method from a semantic route, it is important to note that the meanings of the ordinal relations are not being used to determine mathematical relationships as in the analytic and semi-analytic number-dependent methods. The students do not seem to use the meanings of more and less as inputs to a set of rules that depend on their meaning to make a decision. Instead, they seem to be using a general idea of compensation, fitting a relation between the quantities into a compensation schema, that trades on the verbal description of the magnitudes of these quantities.

It may be going too far to dignify the compensation method with a description as a principle. Instead, it seems that the semantics provides a configuration of the right kinds of elements for the application of this general rule, and the students never bother to cognize the relations between these elements. This claim is consistent with the observation that the compensation idea is applied inappropriately in some cases, and correctly in others. This suggests that the semantics of the quantities are not used directly in the application of the rule (or that the kids have some incorrect rules).

The final non-analytic method was called difference. Sometimes the students decided that a pair of expressions were not equivalent because there were distinct differences between the two expressions. In some cases it appeared to be more like a default method, used when the other methods could not be applied. In these cases, their statements were a listing of differences rather coordinating them as they did with the other methods. Indeed, one subject mentioned how some pairs of expressions were "all messed up" and that there was no "pattern" as there were in other pairs. The decision rule used here seems roughly of the form: If there are differences between the two expressions, then they are not equivalent. By contrast, there were two instances of the difference method that did not appear to be defaults and are very suggestive of possible conceptions that students hold about the structure of arithmetic expressions. We present the instances here; the theoretical issue is taken up in the General Discussion. In the first case, Joanie (Case 14) encountered the expression 685 - 492 + 947 and had to compare it with 947 - 685 + 492 and 947 - 492 + 685. This was the first time in the interview that she had encountered an expression that did not start with the same

number, which was also the largest absolute value. Her complete protocol was brief: "That goes in a separate pile because the number you start out with is lower than in this pile." This is a difference method because she is noting a difference between these two expressions, and does not seem to examine the rest of the expressions.

A similar case occurred when Mary encountered the same situation. She eventually produced a semi-analytic solution, but her original analysis was that the expression beginning with 685 was not equivalent to the other two because:

Case 26: 685 - 492 + 947 947 - 492 + 685

> You start out with 685 [emphasis added], and you take away 492 and you didn't take away 492 in any of the other ones, except [947 - 492 + 685], but you didn't take it from 685.

Mary also seems to note that the expression starts with a different number, and places an importance on the fact that the same pair of numbers were not subtracted from each other. We have observed comparable cases with other students.

Judging Methods for the 12's

There are at least three specific reasons for why the 12's are analyzed separately. First, it was difficult to classify the methods used on the 12's problem set using the concepts developed above. In many cases the students did not state their reasons with the detail found for the other two sets. For equivalence judgments, they reported surface differences between the expressions as not mattering, or described transformations that would reveal the equivalence of two expressions. For judgments of nonequivalence, many times, the students simply reported the discrepancy that led to this judgment. Consequently, it is difficult to determine empirically whether

students were using analytic procedures, difference rules, and so forth. Second, in several cases, the students calculated or did partial calculations for some of the expressions. Frequently, this was done to check the correctness of their analysis. However, it may also have enabled them to generate more sophisticated explanations about why a pair of expressions were equivalent or not. Third, the numbers in the 12's expressions were small, and it may have been easier to form a mental representation that was more difficult to articulate, in contrast to the other two sets in which the students had to construct their arguments more consciously.

Despite these difficulties, we want to describe the performance on two comparisons. The first comparison the students had to make in the entire interview was between 12 - 2 + 6 and 12 + 2 + 6. Of course, they all produced a correct answer. What is more interesting is the diversity of descriptions they produced in giving this answer, most of them consistent with a syntactic, number-independent approach. They are suggestive of subtle differences in how the students are conceiving and describing why these two expressions are not equivalent. This will not be developed here because of the inadequacy of the data, but many of them should be appar

Stella reported that 12 - 2 and 12 + 2 are different. Note that she uses the word different which contrasts with her discussion of the 648 set where she talks about more and less. Nick said that there was a subtracting in one and an adding in another. Both of these are suggestive of a binary analysis. Mary reported that in one expression you are taking away and then adding, while in the other, you are adding both times. This seems more suggestive of a unary analysis and

possibly even a semantic one. Joanie noted that one expression has two pluses and the other has a plus and a minus. Rudolf noted that one expression had a minus so it was different. These last two focus on the operators. Notice the absence of talk about more and less. This was generally true for all the comparisons in the 12's set, which is consistent with our suggestion that number-independent methods were used.

These protocols illustrate that even for one of the simplest equivalence-judging problems, we find protocols indicative of a variety of approaches for reaching the same conclusion.

The second comparison was between 12 - 2 + 6 and 12 + 6 - 2. The students provided sophisticated answers here. Four of the five who made explicit statements said something to the effect that the numbers were in a different order, but that it doesn't matter (see Cases 3, 4, 7, and 8). There appeared to be a binary version in which the students noted that 12 + 6 was equivalent and then two was subtracted (Case 8). There also appeared to be a unary version in which the 12 is treated as a start quantity and that the students recognized that it did not matter when the two was subtracted and the six added, just that it happened at some point (Cases 3, 4, 7). This is taken as evidence that the students are capable of understanding the idea of adding and subtracting respective quantities from a start quantity will result in equivalent expressions regardless of the order in which the quantities are added and subtracted.

General Characteristics of Judging Methods

Having set forth the different judging methods we observed, we now turn to some general observations about applications of these

methods. Three issues are addressed: (a) the extent to which number-independent and analytic judging methods are used, (b) the kinds of errors made, and (c) the use of different judging methods across and within subjects.

The first issue examines the extent to which the students were using number-independent methods in their solution efforts. Each of the problems could be solved with number-independent methods if one were to use unary comparisons. Because the students did not usually pursue such a method, let's consider only those cases in which binary, number-independent comparisons can be made. Most of these involve only commutative transformations. Table 5 contains the five pairs that occurred in the two sets. Performance on these five pairs is summarized in Table 6.

Insert Table 5 about here

Insert Table 6 about here

Students compared these expressions directly for a little over half of the possible opportunities for such comparisons. Of these comparisons, only slightly over half were number-independent comparisons. And of these number-independent comparisons, about one-third were correct. A slightly more liberal tabulating criterion can be used, but it does not change the basic trend. The pair of expressions 947 - 685 + 492 and 947 - 492 + 685 can be included in the analysis because they have a pair of numbers that are equivalent under

commutativity. However, in the context of these two expressions, it is not possible to maintain overall equivalence in an expression, while commuting the binary pair. We can include this case because it is an opportunity in which students could use an number-independent approach, even if incorrect. Table 6 shows a slight proportional increase in the extent to which the students used number-independent methods, but still we do not find that number-independent methods are predominating.

While there is not strong evidence that the students are inclined to using number-independent methods, another way to view their performance is whether they are appealing to analytic methods. When we examine how frequently students used analytic approaches compared to non-analytic approaches, then a slightly more encouraging picture emerges. Table 7 shows that students found analytic solutions about 60% of the time. The non-analytic solutions were evenly distributed among the semi-analytic, compensation, and difference methods (see Table 4).

Insert Table 7 about here

There were individual differences in the observed use of analytic or non-analytic approaches reported in Table 7. It is tempting to speculate that these differences may reflect differences in cognitive style. However, no conclusions are offered because of the following considerations. In some cases, students used a compensation method, which would be counted as non-analytic. However, if they had used the semantics of the ordinal values, as Rudolf did (see Case 2), then they

would produce an analytic solution. Also, in some cases, the students simply did not know what to do, perforce offering a non-analytic solution. For now, we just want to note that students are producing a considerable number of analytic solutions, and suggest that the non-analytic ones may represent the absence of adequate methods rather than the existence of different conceptions.

The second issue concerns the kinds of errors made in the judging performance. The errors can be separated into two kinds. They are violations of algebraic properties, and the non-analytic methods. The only violation of algebraic properties involved subtraction. In one case, a student seemed to invert a binary pair involving subtraction. He asserted that $\underline{a - b}$ equals $\underline{b - a}$ (see Case 21). In a related case, students were asserting that inverted pairs of nested binary expression were equivalent; that is, $\underline{q - a}$ equals $\underline{a - q}$, where $\underline{q}$ is a binary pair (see Case 20). And finally, the backwards method may reflect an assumption that subtraction can be performed in either direction. In short, it seems that some of the students are willing to view inverted subtraction as equivalent. The non-analytic methods are errors in the sense that they cannot reliably produce answers. The compensation method is striking because of the confidence with which the students asserted their answers. Here the problem is not with parsing nor with their knowledge of mathematical properties. For both the compensation and the semi-analytic methods, the students are simply using methods that are not generally effective or which do not use mathematical properties directly such as the semantics of ordinal values.

The third issues concerns general conclusions that can be extracted from the analysis of the judging methods. The observed performance supports the following three hypotheses.

1. Students are not uniform in the range of judging methods they use.

There were 6 possible comparisons in the 947's set and 10 possible comparisons in the 648's set. Here is a count of the number of different methods used by the students across these two sets: Nick (5), Stella (4), Rudolf (2), Mary (4), Joanie (4).

A good illustration of this point comes from Stella comparing a sequence of expressions in the 648's set. In the first comparison, she did a compensation. Her protocol is presented above in Case 15. Then on the next expression, she used an analytic rule based on the semantics of ordinal quantities. A comparable example was given in Case 22. On the following expression she used a semi-analytic method comparable to another semi-analytic method of hers shown in Case 13.

A second illustration shows how a student can use a method that seems to represent the numbers as quantities and evaluate their effects, but then shortly thereafter, appeal to a general principle without using the ideas of quantity just displayed. Mary seemed to evaluate the 12's expressions using an idea along the lines of adding and subtracting numbers from a start quantity. One example was given in Case 4. In a comparable example, she explained why $\underline{12 - 2 + 6}$ is not equivalent to $\underline{12 + 2 - 6}$, because you take away less in one and take away more in the other. This performance contrasts with her performance in the 947's set. In Case 5, Mary seems to give a compensation analysis that mentions ideas of taking away different

amounts as found in the 12's examples. Now, however, she does not seem to use the same idea of taking away different amounts as found in the 12's example, instead appealing to a more general principle. Also, the pair of expressions that she compared in Case 4 are isomorphic to a pair in the 947's (see Case 12). For the 12's she used her semantic method. She never compared the isomorphic pair in the 947's because she used a backwards method to equate it with another expression (see Case 6). A comparable analysis for the same expressions can be shown for Joanie.

The specific reasons for this latter phenomenon are not apparent yet, but a worthy candidate for exploration is the difference between the problem-solving representations of the pairs of expressions. For the 12's, the students may have been able to form appropriate, meaningful relations between the quantities, while with the larger numbers, they resorted to methods that were more syntactic, and number-dependent.

2. Uniform methods are not used for comparing a particular pair of expressions.

For the 947's set, there were six possible comparisons. On five of the six, two or more different methods were used across the collection of students, even for two of the comparisons that only had two students responding. Similarly, there were eight pairs of expressions in the 648's set for which two or more students made comparisons. Six of these eight had at least two different methods used.

Two specific examples illustrate this general point. While three of the five students gave a compensation argument for a pair of 648 expressions (see Cases 15 and 16), at least one student was able to give a correct, analytic argument (see Case 2). Along the same lines, on the pair that was labeled the binary error in the parsing analysis, one sees that some students used an number-independent approach (see Cases 9 and 11), while others used a compensation approach (see Cases 5 and 10).

3. Changing the context can affect the judging method used on an expression.

Frequencies are not provided here because it is difficult to establish what should be counted as a change of method. For example, many of the number-dependent methods may be motivated by the same basic mental procedure, but because of structural features of the expressions the student will produce an analytic, a semi-analytic, a compensation, or a difference method. Consequently, we will simply present three illustrations of this point that do not seem so ambiguous. The point to be established here is that given a particular expression, the method of judging will depend of the expression against which it is being compared.

The first illustration is the contrast in Rudolf's performance with the expression 685 - 492 + 947. When he first had to compare it with 947 - 685 + 492 and 947 - 492 + 685, he asserted that these it was not equivalent because it was negative "Because it would be this [492] plus this [947], which equals more than 685. On the immediately succeeding expression, 947 + 492 - 685, he seemed to ignore what he had just claimed about the first expression, and gave an analytic argument.

At first he thought that they were equivalent, forming an equivalent binary pair with the 947 and the 492. However, he then remembered that subtraction was not commutative. He wrote on a piece of paper 1 - 2 and 2 - 1, explaining that at first he thought that he could do it either way, since this was what one can do with plus, but then remembered that you can't do that. This is a clear example of an number-independent, binary method, and is distinctly different from what he had just done with the same expression compared against two other expressions.

The second illustration is Nick's performance on the expression 648 + 873 - 597. Compared against one expression, he produced a compensation (see Case 16 for a comparable protocol). Compared against another he produced an analytic solution (see Case 22 for a comparable protocol), and compared against a third he produced an number-independent solution (Case 20). While the first two solutions may have been generated by the same process, it is clear that he has produced a different kind of solution in the last case, even though he is always judging the same expression in these three cases.

The third illustration is on Stella's performance on the 947's set. She produced a binary error using an number-independent method (see Case 9 for a comparable example). Then comparing one of these expressions with 685 - 492 + 947, she used a left-to-right binary parse and a semi-analytic method (Case 13). Finally, in comparing the 685 expression with another expression, she used an number-independent method, but with a different binary parsing (Case 1). This illustration shows that the judging method used for a particular expression depends on the expression against which it is being judged.

General Discussion

The major accomplishment of this paper is to describe the parsing and judging methods that some students use to judge the equivalence of three-term arithmetic expressions. We take these methods to reflect the students' capabilities for parsing and judging rather than the limits of their abilities. What is especially attractive to us is that this knowledge seems to stand beyond any particular procedural arithmetic task. It seems to be part of the general conceptual knowledge that children have available for reasoning about arithmetic expressions.

We are pleased by the descriptions of these parsing and judging methods because we see them as a descriptive foundation for developing a more general and formal account of students' structural knowledge about arithmetic expressions. Three main points clearly emerge concerning the students' knowledge and application of these methods. We believe that a satisfactory model of prealgebra students' knowledge of the structure of arithmetic expressions must accommodate these points. First, there are a number of different parsing and judging methods that each student uses, even with the same expression. Second, students are not limited to using a particular kind of parsing or judging method. In fact, they sometimes asserted that certain things had to be true, but then violated these assertions by using different methods. The flexibility with which the students adopt different approaches in handling these problems serves to motivate a strong speculation about the nature of their knowledge, namely, that the students are not rigid and limited in their approaches to understanding the structure of arithmetic expressions. Third, there is evidence that students are able to work with arithmetic as

transformations of quantities, as well as ideas of compensation and ordinal semantics. These three conceptions are all abstract structural concepts that can serve to organize a person's understanding about the structure of an arithmetic expression.

We acknowledge that some of the observed methods may not be well-developed in the sense that the students are not confident about the correctness of them, or that they have been generated in response to the task they were facing. However, for the present analysis, it is not important whether the students were confident about the correctness of their ideas. The observed methods were applied with some consistency, on more than one comparison, by each student. This fact supports the validity of the existence of these different methods. The uncertainty we take to reflect another problem, not addressed here, which is that the students are not usually able to differentiate between the mathematically correct methods and the other methods that they use. This latter claim is grounded on the observation that students asserted particular incorrect analyses with some confidence, and maintained them even when the interviewer provided some mild challenge or suggestion that they could change their answer if desired.

General Conclusions About Parsing

We want to emphasize that the general conclusions just stated support the suggestion that we cannot examine parsing of an arithmetic expression as though it were a uniform process. A comparable argument can be made analytically for algebra. Consider the difference in how one parses an expression when one is trying to simplify it in the context of an equation and when one is trying to factor that expression. These cases suggest that when analyzing parsing methods,

we must consider the task being performed and the context in which it is being done.

A second issue concerns mathematical conventions. The sequence of operations for the expression a - b + c can be interpreted in two ways. We saw that students tended to make an interpretation that violated mathematical conventions. It is important to note that a convention is at issue here, and not a mathematical concept. These students have not been exposed to this convention, and their violations of it suggest that they will not acquire it spontaneously. However, the acquisition of this convention provides a person with an important structural relation between the minus sign in front of the b and in the rest of the expression.

General Conclusions about Judging

There are many issues to be developed here about origins of these different methods, their relative effectiveness, methods of helping students to eliminate or modify certain methods. However, we will only reiterate the point made about the parsing. There are a number of different methods that students have and use, even with the same expression and that we must not view the understanding of structure of expressions as a singular process.

Consistent with the claim that the students have a number of judging methods at a structural level of analysis, we also want to claim that they may have both syntactic and semantic implementations of these methods. The semantic knowledge can be used to justify the syntactic rules. These two different methods may help to clarify the differences between rote and meaningful learning. A rote learner would just have the syntactic form and would not be able to generate

new procedures. By contrast, the semantic form of the judging methods would provide the foundation of applying these concepts (Greeno, 1983).

In discussing the concepts used for the judging analysis, we mentioned a third possibility of an embodiment of the structure of the judging methods. We call it knowledge of arithmetic outcomes. It is a middle ground between a pure syntactic approach and a semantic approach. In this approach, the numbers themselves are the meaningful elements, with their meaning based on viewing the numbers as an asymmetric, transitive ordering. Now, one can reason about more and less, but instead of applying it to physical quantities (as the semantic approach does), it would be applied to the relative position in the ordering of the numbers. This notion needs to be developed further, but it seems to capture our intuition about how we reason about these problems.

Boundary Conditions

This analysis of structural knowledge was conducted on only one task that could be used. Other tasks such as rearranging expressions while preserving equivalence may tap additional important assumptions that students hold about the structure of arithmetic expressions. For example, the two protocols presented under the difference judging method are suggestive of the possibility that some students view the first number of an expression as an important determinant of the equivalence of two expressions (Joanie, Case 14), and that equivalence depends on the same numbers be added and subtracted from each other (Case 26). We saw comparable cases among other students, and this may be worthy of systematic exploration. It should be noted however that like the other methods, this concept does not seem to limit the

students. Joanie was able to reason about expressions that did not have the same start number (Case 16).

The analyses presented here have established the existence of different parsing and judging methods, and noted the range and flexibility of their used by individual students. We have not addressed the question of the conditions under which these different methods are used nor the specific boundary conditions on the particular parses and judging methods that were observed. Among other things, answers to this question should help to clarify reasons for the apparent differences in the judging methods between the one-digit and the three-digit problems sets, and why some problems are handled with a compensation method, while others are handled correctly with a analytic, number-dependent method.

Educational Implications

There are three points to be made. First, we think the most important implication of these results is that they provide a foundation for further scientific investigation into the nature of arithmetic knowledge. The analysis presented in this paper suggests that knowledge of the structure of arithmetic expressions is multi-faceted, and has offered a first attempt at specifying the facets. Much scientific work remains in characterizing this knowledge more explicitly and completely, examining how it is acquired, the degree to which it is coordinated, and the conditions for extension or modification of this knowledge. Such a knowledge base can serve to underpin educational efforts.

Second, the analyses have revealed two conceptual weaknesses that could probably be addressed directly and immediately in the present curriculum. The first is that subtraction cannot be commuted. This can be taught by computing the effect of inverting subtraction. This method was usually effective for communicating this point to students in some pilot instructional studies we conducted. The second conceptual weakness is the canonical interpretation of expressions from a left-to-right form when mixed operations are present. Students do not encounter such expressions in arithmetic, but it may be worthwhile to introduce this issue into the curriculum earlier than algebra, especially because instruction on this point should probably involve physical models and may help to provide a better conceptual understanding of arithmetic.

The third point concerns the implications of this work for the design of arithmetic instruction. A recent report by the National Council of Teachers of Mathematics (1981) notes widespread support for the notion that the concept of basic skills must encompass more than computational facility. The report also noted that 70% of the professionals and laypeople they interviewed expressed their belief in the importance of including instruction in the elementary curriculum concerning generalizations about number patterns. The task we have examined is not in the curriculum at present, but it provides a task that may help children to develop some generalizations about number patterns. The analysis provides some guidance for how it might be used to develop specific mathematical concepts. Specific developments are beyond the scope of the work presented here, so we limit our remarks to the following general point.

If one were trying to develop a curriculum that would foster a conceptual understanding of arithmetic, then the present analysis provides an important hypothesis concerning the point of departure for developing relevant instructional materials. It seems that by the sixth grade, children may already have many of the kinds of concepts they need to reason about the conceptual structure of arithmetic*. Thus, instructional efforts could concentrate on developing appropriate constraints on these ideas, as opposed to the much more difficult objective of communicating the basic ideas in the first place. If we were to pursue this hypothesis in an instructional experiment, we would try to make the different parsing and judging methods described here explicit to the students, and help develop an understanding of the advantages and disadvantages of different ones and the reasons for why some are correct and some are not. We have done this informally using physical models. We found that students need to see the computational consequences of the different methods before they are willing to accept the analytic arguments.

*This hypothesis must be tempered by the fact that above-average children were examined, hence it is important that the hypothesis be tested on a more representative sample of students.

References

Carpenter, P. A., & Just, M. A. (1975). Sentence comprehension: A psychological processing model of verification. Psychological Review, 82, 45-73.

Freeman, D. J., Kuhs, T. M., Porter, A. C., Knappen, L. B., Floden, R. E., Schmidt, W. H., & Schwille, J. R. (1983). Do textbooks and tests define a national curriculum in elementary school mathematics? Elementary School Journal, 33, 501-514.

Greeno, J. G. (1983). Forms of understanding in mathematical problem solving. In S. G. Paris, G. M. Olson, & H. W. Stevenson (Eds.), Learning and motivation in the classroom (pp. 83-111). Hillsdale, NJ: Erlbaum.

Greeno, J. G., Riley, M. S., & Gelman, R. (1984). Conceptual competence and children's counting. Cognitive Psychology, 16, 94-143.

Kieran, C. (in press). Algebra. In T. Husen & T. N. Postlethwaite (Eds.), International encyclopedia of education: Research and studies. Oxford: Pergamon.

National Advisory Committee on Mathematics (1975). Overview and analysis of school mathematics: grades K-12. Washington, D.C.: Conference Board on the Mathematical Sciences.

National Council of Teachers of Mathematics. (1981). Priorities in school mathematics: Executive summary of the PRISM project. Reston, VA: Author.

Weaver, J. F. (1982). Interpretations of number operations and symbolic representations of addition and subtraction. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), Addition and subtraction: A cognitive perspective (pp. 60-66). Hillsdale, NJ: Erlbaum.

Table 1

Problem Sets Given to Students

For Equivalence Judging

12's set
12 - 2 + 6
12 + 2 + 6
12 + 2 - 6
12 + 6 - 2

947's set
947 - 685 + 492
947 - 492 + 685
685 - 492 + 947
947 + 492 - 685

648's set
648 + 873 - 597
873 + 597 - 648
597 + 648 - 873
648 - 597 + 873
597 - 648 + 873

Table 2

Correctness of Equivalence Judging

for the Three Problem Sets

	Problem Set								
Student	12's			947's			648's		
	C	IC	NC	C	IC	NC	C	IC	NC
Nick	3	0	3	3	2	1	5	4	1
Mary	4	0	2	3	3	0	3	2	5
John	3	0	3	2	4	0	1	4	5
Joanie	6	0	0	3	3	0	5	2	0
Rudolf	4	0	2	3	3	0	8	2	0
Stella	4	0	2	1	3	0	3	3	4
Total	24	0	12	15	16	1	25	17	18

Note. C =Correct; IC=Incorrect;

NC=No comment, there was no explicit comparison.

Table 3

Frequency of Observed Parsing Categories for the Three Problem Sets

	Problem Set		
Parsing Type	12's	947's	648's
Whole			
Binary	2	11	22
Unary	6	0	1
Left-to-right	1	8	3
Backwards	0	2	1
Partial			
Operators	4	0	1
First additive pair	2	0	3
Subtracting one term	2	0	2
First number	0	2	0
Other			
Calculated	1	0	0
Uninterpreted	1	0	0
Total	19	23	33

Table 4

Frequency of Observed Judging Methods for the 947's and 648's Problem Sets

	Problem Set	
Judging Methods	947's	648's
Analytic		
Number-independent	8	6
Number-dependent	2	12
Subtotal	10	18
Non-analytic		
Semi-analytic	2	5
Compensation	4	4
Difference	5	2
Subtotal	11	11
Grand Total	21	29

Table 5

Pairs of Expressions that Have Number-Independent Binary Solutions

685 - 492 + 947	947 + 492 - 685
648 + 873 - 597	597 - 648 + 873
873 + 597 - 648	648 - 597 + 873
873 + 597 - 648	597 - 648 + 873
648 - 597 + 873	597 - 648 + 873

Table 6

Frequency of Number-Independent Judging Rules Used for Appropriate Pairs of Expressions in the 947's and 648's Problem Sets

	Classification Criterion	
Response	Strict	Liberal
Correct Number-independent	3	3
Number-independent(a)	8	13
Total(b)	14	19
Possible(c)	25	30

Note: (a) Number-independent refers to the number of algebraic judging rules that were used among the Total comparisons made.

(b) Total refers to the number of comparisons that students actually made.

(c) Possible refers to the number of comparisons that could have been observed for the relevant pairs of expressions.

Table 7

Frequency of Analytic and Non-analytic Judging Rules for the 947's and 648's Problem Sets

Student	Analytic	Non-analytic
Nick	7	5
Mary	2	6
John	1	0
Joanie	3	7
Rudolf	10	0
Stella	5	4
Total	28	22

Rudolf

(12) (- 2) (+ 6) (12) (+ 6) (- 2)

Stella

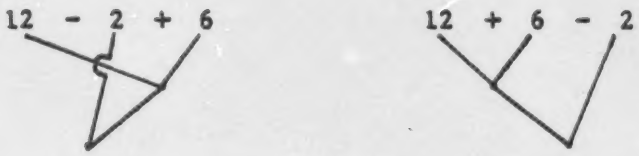


Figure 1. Different parses of same pair of expressions in 12's set.

Rudolf

Joanie

947 - 685 + 492 947 - 492 + 685

Figure 2. Different parses of same pair of expressions in 947's set.

Stella

Mary

(685) (- 492) (+ 947) (947) (+ 492) (- 685)

Figure 3. Different parses of same pair of expressions in 947's set.

Nick

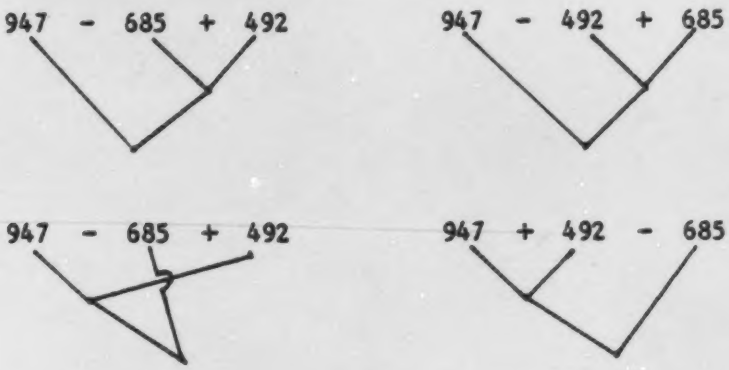


Figure 4. Nick's different parses of the same expression in 947's set.

Stella

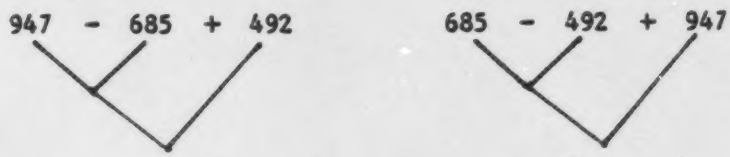
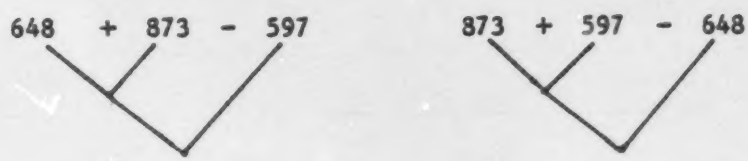


Figure 5. Stella's different parses of the same expression in 947's set.

Stella



Joanie

Figure 6. Two different parses of same pair of expressions in 648's set.

Nick

873 + 597 - 648 648 - 597 + 873

Rudolf

Mary

873 + 597 - 648 648 - 597 + 873

Figure 7. Three different parses of same pair of expressions in 648's set.

Nick

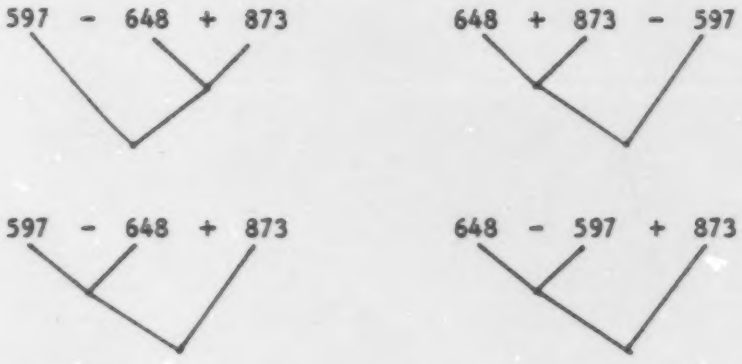


Figure 8. Nick's different parses of the same expression in 648's set.

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